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A Remark on Orthonormal Bases of Continuous Functions in a Hilbert Space

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A REMARK ON ORTHONORMAL BASES OF CONTINUOUS FUNCTIONS IN A HILBERT SPACE

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Group 66

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Abstract

It is shown that an orthonormal set of continuous functions on a finite interval can always be completed by the addition of continuous functions if it is a finite set but cannot always be so completed if it is an infinite set.

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A Remark on Orthonormal Bases of
Continuous Functions in a Hilbert Space

The following problem has been proposed: let H be the Hilbert space of square integrable functions on a finite interval I and let (φ_i) be an orthonormal set of continuous functions in H — when can (φ_i) be extended to a complete orthonormal set of continuous functions? This problem occurs under certain circumstances when approximating white noise by sums $\sum \theta_i \varphi_i(t)$ where the θ_i are independent Gaussian random variables.

In this note we prove that (φ_i) can be so extended if it is a finite set and present an example to show that (φ_i) cannot always be extended if it is infinite. Both the proof and the example apply equally well if continuity is replaced by n-times differentiability.

Theorem If $\varphi_1, \ldots, \varphi_n$ is an orthonormal set of continuous functions then continuous functions $\varphi_{n+1}, \varphi_{n+2}, \ldots$ can be found such that φ_i, $i = 1, 2, \ldots$ is a complete orthonormal set.

Proof Let (ψ_i) be a complete orthonormal set of continuous functions (e. g., the trigonometric functions). Set

$$\varphi_{n+1} = \alpha_1(\psi_1 - \sum_{j=1}^{n} (\psi_1, \varphi_j)\varphi_j)$$

where α_1 is 0 if ψ_1 is a linear combination of the φ_i's and is chosen to normalize φ_{n+1} otherwise. Continuing in this way, i. e., setting

$$\varphi_{n+k+1} = \alpha_{k+1}(\psi_{k+1} - \sum_{j=1}^{n+k} (\psi_{k+1}, \varphi_j)\varphi_j),$$

deleting the φ_{n+k} with $\alpha_k = 0$ and then renumbering gives the desired sequence.

We will need the following lemma in constructing the example.

Lemma On any finite interval I with end point a there exists a complete orthonormal set $\varphi_o, \varphi_1, \ldots$ satisfying

(1) $\varphi_o = 1$

(2) φ_i is continuous

(3) $\varphi_i(a) = 0$, if $i > 0$.

Proof There exist continuous functions ψ_i such that $\varphi_o, \psi_1, \psi_2, \ldots$ is a complete orthonormal set. Let η be a continuous function with $\eta(a) = 0$ and $(\varphi_o, \eta) = 1$. Given any continuous function ξ and any $\varepsilon > 0$ we can, by modifying ξ in a sufficiently small neighborhood of a, construct a continuous function ξ' with $\xi'(a) = 0$ and $\| \xi - \xi' \| < \varepsilon$. Then $\xi'' = \xi' - (\varphi_o, \xi')\eta$ is continuous and satisfies

(i) $\xi''(a) = 0$

(ii) $(\xi'', \varphi_o) = 0$

(iii) $\| \xi - \xi'' \| \leq \| \xi - \xi' \| + |(\varphi_o, \xi')| \, \| \eta \|$

$$\leq \varepsilon(1 + \| \eta \|) + (\varphi_o, \xi) \| \eta \| .$$

In particular taking $\xi = \psi_i$ so that the last term vanishes and choosing $\varepsilon = 2^{-k}/(1 + \| \eta \|)$ we can construct a continuous function $\psi_{i,k}$ with

(iv) $\psi_{i,k}(a) = 0$

(v) $(\psi_{i,k}, \varphi_o) = 0$

(vi) $\| \psi_i - \psi_{i,k} \| \le 2^{-k}$.

Now when we apply the Gram-Schmidt procedure to the sequence $\varphi_o, \psi_{1,1}, \psi_{1,2}, \psi_{2,1}, \psi_{2,2} \ldots$ none of the linear combinations after the first involve φ_o by (v) so they all vanish at a. The resulting orthonormal sequence approximates the ψ_i's arbitrarily closely, hence is complete, and hence satisfies the requirements of the lemma.

We will now construct a complete orthonormal set $\psi_1, \ldots$ with ψ_1 discontinuous and all the ψ_i, $i > 1$ continuous. This if we take $\psi_2, \psi_3, \ldots$ for our orthonormal set of continuous functions it can only be completed by adjoining the discontinuous function ψ_1 or $-\psi_1$. We divide the interval I into subintervals I_1 and I_2 at the point a and choose complete orthonormal sets $\varphi_o^i, \varphi_1^i, \ldots$ in I_i according to the previous lemma. Then the set $\psi_1 = \varphi_o^1 - \varphi_o^2$, $\psi_2 = \varphi_o^1 + \varphi_o^2$, $\varphi_1^1, \varphi_1^2, \varphi_2^1, \varphi_2^2, \ldots$ is complete and orthonormal and has ψ_1 as its only discontinuous member.

In the above example $(\varphi_i)^\perp$, the orthogonal complement of the given set of φ's was finite dimensional. An example with infinite dimensional $(\varphi_i)^\perp$ can be constructed by breaking I into three intervals, say $I_1 = [0, a]$, $I_2 = [a, b]$ and $I_3 = [b, 1]$. We will need the following modification of the above lemma.

<u>Lemma</u> There exists a complete orthonormal set $\varphi_1^2, \varphi_2^2, \ldots$ on $[a, b]$ in which each φ_i is a continuous function with $\varphi_i(a) = \varphi_i(b) = 0$.

Proof The proof is similar to that of the preceding lemma except that (iv) is changed to

$$\text{(iv}'\text{)} \qquad \psi_{i,k}(a) = \psi_{i,k}(b) = 0$$

and (v) is dropped.

Now we take sets $(\varphi_o^1, \varphi_1^1, \ldots)$, $(\varphi_1^2, \varphi_2^2, \ldots)$ and $(\varphi_o^3, \varphi_1^3, \ldots)$ such that (φ_i^j) is a complete orthonormal set in I_j, $\varphi_o^1 = 1$ on I_1, $\varphi_o^3 = 1$ on I_3, and $\varphi_i^1(a) = \varphi_i^2(a) = \varphi_i^2(b) = \varphi_i^3(b) = 0$ for $i \geq 1$. We take for our set of continuous orthonormal functions the union of the sets $(\varphi_1^1, \varphi_2^1, \ldots)$ and $(\varphi_1^2, \varphi_2^2, \ldots)$. The orthogonal compliment of this set is all functions of the form $c\varphi_o^1 + f$ where f vanishes outside I_3. No matter how the set is completed it will contain at least one function of the above form with $c \neq 0$ and hence with a discontinuity at a.

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